Locally checkable problems parameterized by mim-width

Carolina Lucía Gonzalez

November 11, 2024

mim-width

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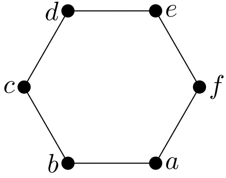

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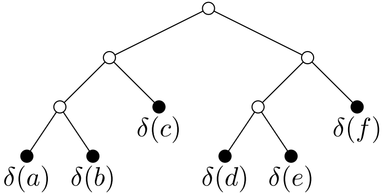

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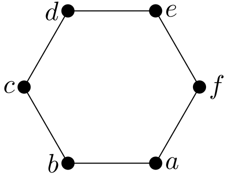

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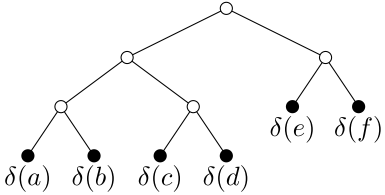

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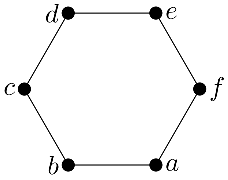

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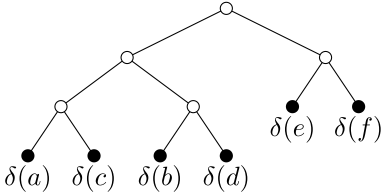

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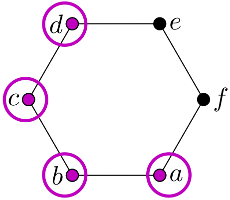

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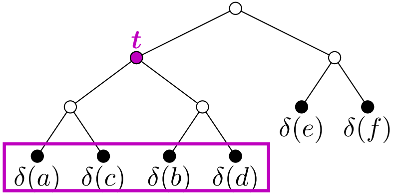
$$\boldsymbol{T_t = \{a, b, c, d\}}$$



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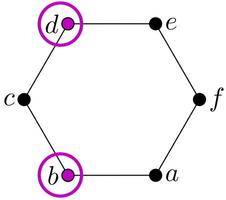

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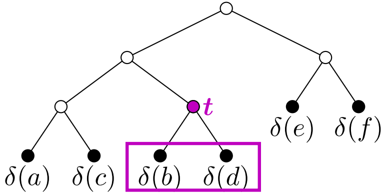

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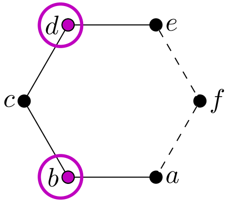

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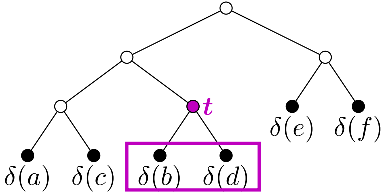

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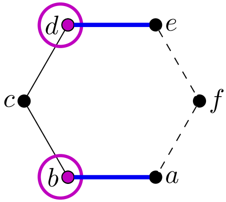

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$\boldsymbol{T_t = \{b, d\}}$

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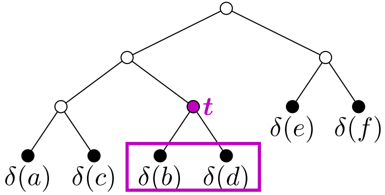

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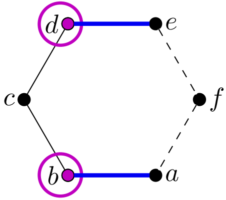

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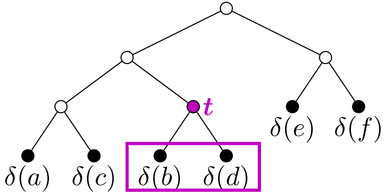

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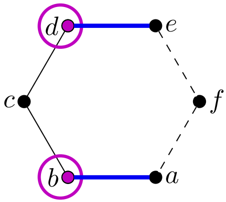
The **mim-width of a graph** G, $\boldsymbol{mimw(G)}$, is the minimum value of $mimw(T, \delta)$ over all binary decomposition trees (T, δ) of G.



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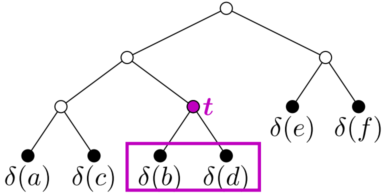

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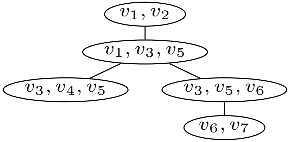
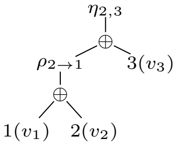
Width parameters

These parameters are associated with useful ways to describe the graph.

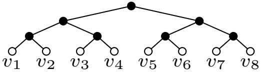
- treewidth: tree-decompositions



- clique-width: clique-width expressions

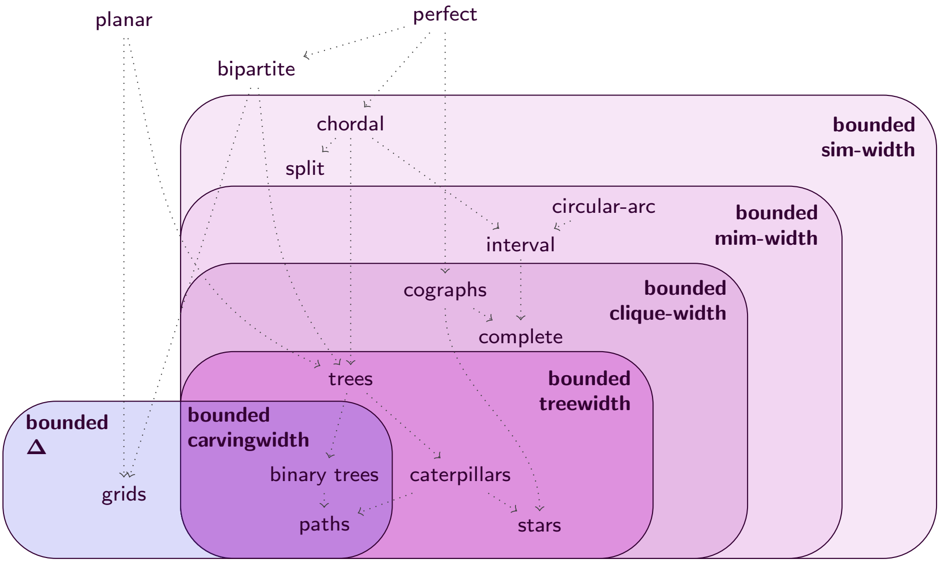


- mim-width: branch decompositions



$\to$ **Dynamic programming** gets along really well with them!

Width parameters



Parameterized complexity

In a **parameterized problem** the input is considered to be two-part: (x, k)

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main input

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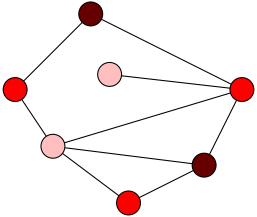
W[1]-hard: XP but *unlikely* to be FPT.

Introductory problems

k-coloring



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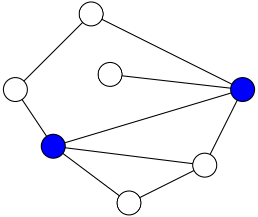
Minimum Dominating Set



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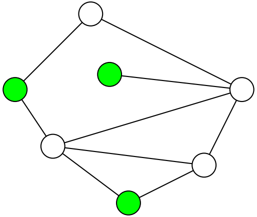

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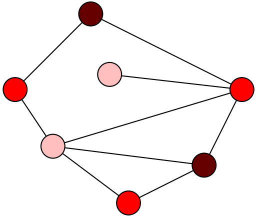

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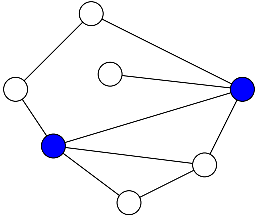

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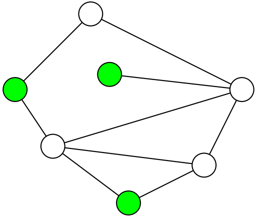

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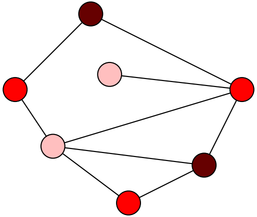

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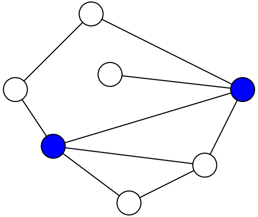

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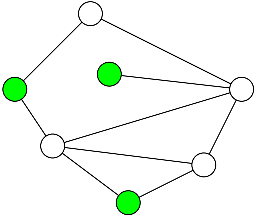

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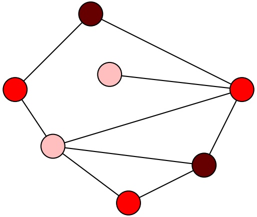

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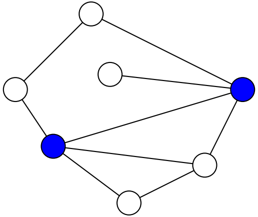

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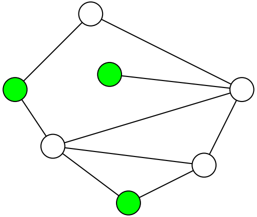

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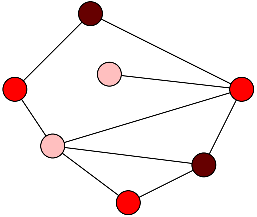

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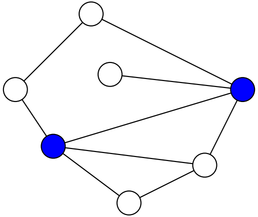

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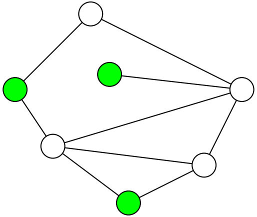

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$\rightarrow$ **Locally checkable problems**

Why do we study locally checkable problems?

- Many well-known graph problems are locally checkable, with real-life applications:
  - Assigning resources (classrooms, processor registers, frequencies)
  - Placing facilities (recycling points, information centers, vending machines)

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- Many well-known graph problems are locally checkable, with real-life applications:
  - Assigning resources (classrooms, processor registers, frequencies)
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- Can we solve these problems efficiently?
  - For which graph classes?
  - How?
  - Can we generalize the ideas?

Related work

- Locally Checkable Vertex Partitioning (LCVP) problems
  [J.A. Telle, 1994]
- Local Condition Composition (LCC) and Edge Condition Composition (ECC) problems
  [H.L. Bodlaender, 1987]
- Monadic Second Order Logic (Courcelle's theorem)
  [B. Courcelle, 1990]
- A&C DN logic
  [B. Bergougnoux, J. Dreier and L. Jaffke, 2022]

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Study the **complexity** of locally checkable problems **in terms of different width parameters** (carvingwidth, treewidth, clique-width and mim-width).

$\rightarrow$ develop efficient algorithms under a set of restrictions

$\rightarrow$ find hard problems that do not satisfy all these restrictions

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The weight c is $\mathrm{w}(c) = \sum_{v \in V(G)} \mathrm{w}_{v,c(v)}$.

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- $\textsc{Colors}$: a set of colors,
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find the minimum weight of an assignment of colors to vertices $c\colon V(G) \to \textsc{Colors}$ such that

$$\mathit{check}(v, c|_{N[v]}) = \textsc{True} \quad \text{for all } v \in V(G).$$

Minimum Dominating Set:

$\textsc{Colors} = \{\mathbf{s}, \overline{\mathrm{s}}\}$

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Maximum Independent Set:

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Order of the weights: $\geq$

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k-coloring:

$\textsc{Colors} = \{1, \ldots, k\}$

$check(v,c) = (\forall u \in N(v).\, c(u) \neq c(v))$

$\mathrm{w}_{v,i} = 0$

Order of the weights: $\leq$

Global properties

A locally checkable problem with **global properties** $\Pi = \{P_1, \ldots, P_k\}$ is a locally checkable problem where it is additionally required that the assignment of colors to vertices c given as an answer satisfies each of the properties in Π, i.e., that $P_i(c) = \textsc{True}$ for each $i \in \{1, \ldots, k\}$.

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For example:

- "the number of vertices of color red or blue is at most 10"
- "there is an odd number of green vertices"

- "the subgraph induced by the set of red vertices is connected"
- "the subgraph induced by the set of blue and green vertices is not connected"

- "the subgraph induced by the set of green vertices is acyclic (i.e. is a forest)"
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  $\longrightarrow$ **Size properties**
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  $\longrightarrow$ **Connectivity properties**
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  $\longrightarrow$ **Acyclicity properties**

Main goal

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We want to obtain efficient algorithms when parameterized by different width measures
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Under which conditions?

Which global properties can we include?

Color-counting local property

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In other words, if there exists a function $check'$ such that, for all $v \in V(G)$ and $c\colon N[v] \to \textsc{Colors}$:

$$\boldsymbol{check(v, c) = check'(v, c(v), \mu_1, \ldots, \mu_q)}$$

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Example: Minimum Dominating Set

- $check'(v, i, \mu_{\mathbf{S}}, \mu_{\overline{\mathrm{S}}}) = (i = \mathbf{S} \vee \mu_{\mathbf{S}} \geq 1)$

d-stable local property

Let d be a positive integer.

We say that $check$ is **d-stable** if it only depends on:

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- the color that v receives, and
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In other words, if $check$ is color-counting and:

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Examples:

- Minimum Dominating Set (1-stable)
- LCVP problems

Locally checkable problems parameterized by mim-width

These results were published in an article together with Felix Mann (2024).

Restrictions on $check$	Restrictions on $\lvert\textsc{Colors}\rvert$	
	Constant	Polynomial
d-stable	XP$^{(1)}$, W[1]-h	para-NP-h
Color-counting	para-NP-h	para-NP-h

(1) given a binary decomposition tree.

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Dynamic programming algorithm using the given binary decomposition tree.

Complexity: $n^{O(wd\mathcal{C})}$

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We may add some global properties:

- Size
- Connectivity

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W[1]-hard example: Minimum Dominating Set

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Examples of para-NP-hard problems: Max-Cut, L(2,1)-labeling

Example: Roman $\{k\}$-domination

Given a graph G, what is the minimum weight $\sum_{v \in V} f(v)$ of a function $f\colon V \to \{0, \ldots, k\}$ such that

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  $\to$ $k + 1$ colors

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  $\rightarrow$ **color-counting, and k-stable**

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$\rightarrow$ XP parameterized by mim-width

A&C DN logic

The **distance neighborhood logic with acyclicity and connectivity constraints** was presented by Bergougnoux, Dreier and Jaffke in 2022.

They proved that every problem expressible by a formula ϕ of constant length (and another constant parameter, namely, $d(\phi)$) in this logic is XP parameterized by the mim-width of a given decomposition.

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Problems that can be modeled by using A&C DN logic formulas of constant length and constant $d(\cdot)$

Problems that can be modeled as locally checkable problems with

- global properties of size, connectivity and acyclicity
- constant number of colors,
- d-stable functions at t distances (for some constants d, t)

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$\equiv$

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- constant number of colors,
- d-stable functions at t distances (for some constants d, t)

The framework (complete)

r-locally checkable problem

Given a graph G and

- $\textsc{Colors}$: a set of colors,
- L_v: for each vertex v, a subset of $\textsc{Colors}$ of permitted colors,
- $\textsc{Labels}$: a set of edge labels,
- ℓ_e: for each edge e, a label in $\textsc{Labels}$,
- $(\textsc{Weights}, \preceq, \oplus)$: a weight set,
- $\mathrm{w}(v, c)$: a weight function (input: vertex v and coloring c of $N^r[v]$, output: a weight),
- $check(v, c)$: check function (input: vertex v and coloring c of $N^r[v]$, output: true/false)

find the minimum weight of a coloring c such that $c(v) \in L_v$ and $check(v, c|_{N^r[v]}) = \textsc{True}\ \forall v \in V$.

Global properties considered:

- Connectivity
- Acyclicity
- Local condition composition

Summary of results

	$\lvert\textsc{Colors}\rvert$	ℓ_e	$check$, w	Global properties
carw	Polynomial	Yes	Computable in polynomial time	Connectivity, acyclicity, local cond. comp.
tw	Polynomial	Yes	Polynomial partial neighborhood system	Conectivity, acyclicity, local cond. comp.
cw	Polynomial	No	Polynomial universal partial neigh. sys.	Local condition composition
mimw	Constant	No	d-stable	Connectivity, acyclicity, local cond. comp.

The results presented throughout this whole thesis generalize those of some previous frameworks (such as LCC and ECC problems [Bodlaender, 1987] and distance LCVP problems [Jaffke, Kwon, Strømme and Telle, 2019]).

The results about the parameterization by mim-width are *almost equivalent* to those obtained by Bergougnoux, Dreier and Jaffke (2022) for A&C DN logic.

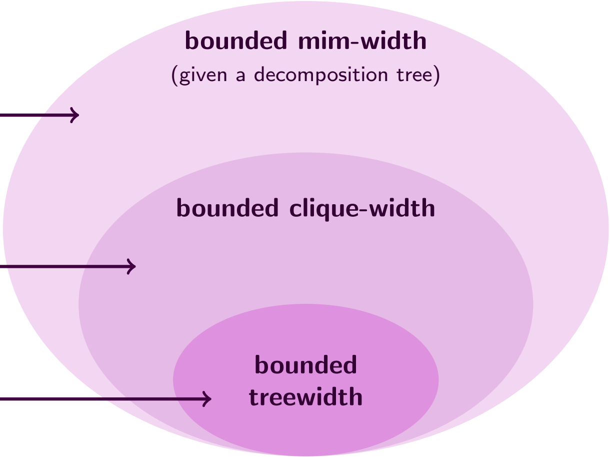
We also included PTAS on planar graphs for some locally checkable problems.

Summary of applications

Roman $\{k\}$-domination$^{(*)}$
Maximum dual domination
k-maximum happy vertices$^{(*)}$
Conflict-free k-coloring
M-GPF

k-community
Max-PDS
Maximum γ-quasi-clique
Signed Roman domination
Global $[k]$-Roman domination
GPF with polynomial k, u

Minimum chromatic violation
Grundy domination$^{(*)}$
Additive coloring



$^{(*)}$**PTAS on planar graphs**

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